THIRUVALLUVAR UNIVERSITY COLLEGE OF ARTS AND SCIENCE

TIRUPATTUR

STUDY MATERIALS

BAMA15C-MATHEMATICS I

For

II YEAR PHYSICS, III SEMESTER

Prepared by

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MATHEMATICS I

Objectives: To Explore the Fundamental Concepts of Mathematics.

UNIT-I: ALGEBRA Partial Fractions - Binomial, Exponential and logarithmic Series (without Proof) - Summation - Simple problems.

UNIT-II: THEORY OF EQUATIONS Polynomial Equations with real Coefficients - Irrational roots - Complex roots- Transformation of equation by increasing or decreasing roots by a constant - Reciprocal equations - Newton's method to find a root approximately - Simple problems.

UNIT-III: MATRICES Symmetric - Skew-Symmetric - Orthogonal and Unitary matrices - Eigen roots and eigen vectors – Cayley - Hamilton theorem (without proof)-Verification and computation of inverse matrix.

UNIT-IV: TRIGONOMETRY Expansions of $sinn\theta$, $cosn\theta$, $sinn\theta$, $cosn\theta$, $tann\theta$ - Expansions of $sin\theta, cos\theta, tan\theta$ in terms of θ.

UNIT-V: DIFFERENTIAL CALCULUS Successive differentiation upto third order, Jacobians -Concepts of polar co-ordinates-Curvature and radius of curvature in Cartesian co-ordinates and in polar co-ordinates.

Recommended Text Books:

P.Duraipandian and S.Udayabaskaran,(1997) Allied Mathematics, Vol. I & II.Muhil Publishers, Chennai.

Chapter-2: Sections 2.1-2.10 (Omit Applications 1 and 2 of 2.7)

Chapter-3: Sections 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7.

Reference Books:

1. P.Balasubramanian and K.G.Subramanian,(1997) Ancillary Mathematics. Vol. I & II. Tata McGraw Hill, New Delhi.

2. S.P.Rajagopalan and R.Sattanathan,(2005) Allied Mathematics .Vol. I & II. VikasPublications, New Delhi.

3. P.R.Vittal (2003) Allied Mathematics .Marghan Publications, Chennai

4. P.Kandasamy, K.Thilagavathy (2003) Allied Mathematics Vol-I, II S.Chand& company Ltd., New Delhi-55.

5. Isaac, Allied Mathematics. New Gamma Publishing House, Palayamkottai.

UNIT I

ALGEBRA

Partial Fractions: Consider an rational expression

$$f(x) = \frac{p(x)}{q(x)}$$

where both $p(x)$ and $q(x)$ are polynomials and the degree of $p(x)$ is smaller than the degree of $q(x)$.

Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator.

•Factor in denominator $ax + b$
Term in partial fraction decomposition $\frac{A}{ax+b}$.

•Factor in denominator $(ax + b)^k$
Term in partial fraction decomposition $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$, $k = 1, 2, 3, \ldots$.

•Factor in denominator $ax^2 + bx + c$
Term in partial fraction decomposition $\frac{Ax+B}{ax^2+bx+c}$.

•Factor in denominator $(ax^2 + bx + c)^k$
Term in partial fraction decomposition $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$, $k = 1, 2, 3, \ldots$

Problem 1. *Evaluate the integral* $\int \frac{3x+11}{x^2-x-6}dx$.

Solution:

$$\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$
$$\frac{3x+11}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$
$$3x + 11 = A(x + 2) + B(x - 3)$$

By substituting $x = 3$ and $x = -2$, we get

$$A = 4, B = -1$$

Therefore,

$$\int \frac{3x + 11}{x^2 - x - 6} dx = \int \left(\frac{4}{x - 3} - \frac{1}{x + 2} \right) dx$$

$$\int \frac{3x + 11}{x^2 - x - 6} dx = 4 \log(x - 3) - \log(x + 2) + c$$

Home Work:

Evaluate the integral $\int \frac{x^2}{x^2-1} dx$.

Problem 2. *Find the coefficient of x^n in $\frac{1}{(1+2x)(1+3x)}$.*

Solution:

$$\frac{1}{(1 + 2x)(1 + 3x)} = \frac{A}{(1 - 2x)} + \frac{B}{(1 + 3x)}$$
$$1 = A(1 + 3x) + B(1 - 2x)$$

Put $x = \frac{1}{2}$, we get

$$1 = A\left(1 + 3\left(\frac{1}{2}\right)\right) + B\left(1 - 2\left(\frac{1}{2}\right)\right)$$
$$1 = A\left(1 + \frac{3}{2}\right)$$
$$1 = A\left(\frac{5}{2}\right)$$
$$A = \frac{2}{5}$$

Put $x = -\frac{1}{3}$, we get

$$1 = B\left(1 - 2\left(-\frac{1}{3}\right)\right)$$
$$1 = B\left(\frac{5}{3}\right)$$
$$B = \frac{3}{5}.$$

Therefore,

$$\begin{aligned}\frac{1}{(1+2x)(1+3x)} &= \frac{2}{5}(1-2x)^{-1} + \frac{3}{5}(1+3x)^{-1} \\ &= \frac{2}{5}\left[1 + 2x + (2x)^2 + (2x)^3 + \cdots + (2x)^n\right] \\ &\quad + \frac{3}{5}\left[1 + (-3x) + (-3x)^2 + (-3x)^3 + \cdots + (-3x)^{-n}\right]\end{aligned}$$

Therefore, coefficient of $x^n = \frac{2}{5}2^n + \frac{3}{5}(-3)^n$

$$\implies x^n = \frac{2^{n+1} + (-1)^n 3^{n+1}}{5}$$

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BINOMIAL SERIES:

The Binomial theorem for positive integral integers

$$(1-x)^{-\frac{p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

(1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
(2) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
(3) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$
(4) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
(5) $(1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$
(6) $(1+x)^{-n} = 1 - \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$
(7) $(1+x)^{n} = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$
(7) $(1-x)^{n} = 1 - \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

Problem 3. *Sum the series*

$$S = 1 + \frac{2}{3}\left(1/2\right) + \frac{2.5}{3.6}\left(1/2\right)^2 + \frac{2.5.8}{3.6.9}\left(1/2\right)^3 + \dots$$

Solution:

The factors in the numerators are in Arithmetic Progression which implies

$$1 + \frac{2}{1!}\left(1/6\right) + \frac{2\cdot 5}{2!}\left(1/6\right)^2 + \frac{2\cdot 5\cdot 8}{3!}\left(1/6\right)^3 + \dots$$

Here,

$$\frac{x}{q} = \frac{1}{6} \implies x = \frac{3}{6} \implies x = \frac{1}{2}.$$

Now,

$$\begin{aligned}(1-x)^{-\frac{p}{q}} &= \left(1-\frac{1}{2}\right)^{-\frac{2}{3}} \\ &= (2)^{-\frac{2}{3}} \\ &= \sqrt[3]{(2)^2} \\ &= \sqrt[3]{4}.\end{aligned}$$

Therefore,

$$S = \sqrt[3]{4}$$

Problem 4. *Prove that*

$$\begin{aligned}1 + n\left(\frac{2n}{1+n}\right) + \frac{n(n+1)}{1.2}\left(\frac{2n}{1+n}\right)^2 + \frac{n(n+1)(n+2)}{1.2.3}\left(\frac{2n}{1+n}\right)^3 + \dots \\ = 1 + n\left(\frac{2n}{1-n}\right) + \frac{n(n-1)}{1.2}\left(\frac{2n}{1-n}\right)^2 + \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{2n}{1-n}\right)^3 + \dots\end{aligned}$$

Proof. Here,

$$p = n,\ q = 1$$

and

$$\frac{x}{q} = \frac{2n}{n+1}$$

$$\implies x = \frac{2n}{n+1}.$$

Now,

$$(1-x)^{-\frac{p}{q}} = \left(1 - \frac{2n}{1+n}\right)^{-n} = \left(\frac{1+n-2n}{1+n}\right)^{-n}$$

Therefore, LHS $= \left(\frac{1-n}{1+n}\right)^{-n}$.

In RHS,

$$p = n, \ q = -1$$

and

$$\frac{x}{q} = \frac{2n}{n-1}$$

$$\implies x = -\left(\frac{2n}{n-1}\right)$$

Now,

$$(1-x)^{-\frac{p}{q}} = \left(1 + \frac{2n}{1-n}\right)^{n} = \left(\frac{1-n+2n}{1-n}\right)^{n} = \left(\frac{1+n}{1-n}\right)^{n}$$

Therefore, RHS $= \left(\frac{1-n}{1+n}\right)^{-n}$. Hence LHS=RHS. □

Problem 5. *Sum the series*

$$S = 1 - \tfrac{1}{5} + \tfrac{1.4}{5.10} - \tfrac{1.4.7}{5.10.15} + \dots$$

Solution:

$$S = 1 + (-1)\left(\tfrac{1}{5}\right) + \tfrac{-1.-4}{2!}\left(\tfrac{1}{5}\right)^2 + \tfrac{-1.-4.-7}{3!}\left(\tfrac{1}{5}\right)^3 + \dots$$

Here,

$$p = -1, q = -3$$

and

$$\frac{x}{q} = \left(\frac{1}{5}\right)$$

$$\implies x = -\left(\frac{3}{5}\right)$$

Therefore,

$$\begin{aligned}(1-x)^{-\frac{p}{q}} &= \left(1+\frac{3}{5}\right)^{-\frac{1}{3}} \\ &= \left(\frac{5+3}{5}\right)^{-\frac{1}{3}} = \left(\frac{8}{5}\right)^{-\frac{1}{3}} \\ &= \left(\frac{5}{8}\right)^{-\frac{1}{3}} \\ &= \sqrt[3]{\frac{5}{8}}\end{aligned}$$

Problem 6. *Sum to* ∞ *or*

$$s_\infty = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots + \infty$$

Solution

$$s_\infty = 1 + \frac{3}{1!}\left(\frac{1}{4}\right) + \frac{3.5}{2!} + \left(\frac{1}{4}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{4}\right)^3 + \cdots + \infty$$

Here,

$$p = 3, q = 2$$

and

$$\begin{aligned}\frac{x}{q} &= \frac{1}{4} \\ \implies x &= \frac{2}{4} \\ \implies x &= \frac{1}{2}\end{aligned}$$

Now,

$$\begin{aligned}(1-x)^{-\frac{p}{q}} = \left(1-\frac{1}{2}\right)^{-\frac{3}{2}} &= \left(\frac{2-1}{2}\right)^{-\frac{3}{2}} = \left(\tfrac{1}{2}\right)^{-\frac{3}{2}} \\ &= \left(2^3\right)^{\frac{1}{2}} = \sqrt{2^3} = \sqrt{8}\end{aligned}$$

Problem 7.

$$s_\infty = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots \infty$$

$$1 + s_\infty = 1 + \frac{1}{1!}\left(\frac{1}{3}\right) + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + \ldots$$

Here,

$$p = 1, q = 2$$

and

$$\frac{x}{q} = \frac{1}{3}$$

$$\implies x = \frac{2}{3}.$$

Therefore,

$$(1-x)^{-\frac{p}{q}} = \left(1 - \frac{2}{3}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{1}{3}\right)^{-\frac{1}{2}}$$

$$= (3)^{\frac{1}{2}}$$

$$= \sqrt{3}$$

$$S_\infty = \sqrt{3} - 1$$

Problem 8. *Show that x is a small that*

$$\sqrt{x^2+4} - \sqrt{x^2+1}$$

is

$$1 - \frac{x^2}{4} + \frac{7x^4}{64}$$

Solution

$$\sqrt{x^2+4} = \sqrt{4\left(1 + \tfrac{x^2}{4}\right)}$$

$$= 2\left(1 + \tfrac{x^2}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1 + \tfrac{1}{2}\left(\tfrac{x^2}{4}\right) + \tfrac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}\left(\tfrac{x^2}{4}\right)^2\right]$$
$$= 2\left[1 + \tfrac{x^2}{8} - \tfrac{1}{8}\tfrac{x^4}{16}\right]$$

Now,

$$\sqrt{x^2+1} = \sqrt{(1+x^2)}$$
$$= (1+x^2)^{\frac{1}{2}}$$
$$= 1 + \tfrac{x^2}{2} - \tfrac{x^4}{8}$$

Now,

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (2-1) + \left(\tfrac{1}{4} - \tfrac{1}{2}\right)x^2 + \left(\tfrac{-1}{64} + \tfrac{1}{8}\right)x^4$$
$$= 1 + \left(\tfrac{1-2}{4}\right)x^2 + \left(\tfrac{-1+8}{64}\right)x^4$$
$$= 1 - \tfrac{x^2}{4} + \tfrac{7x^4}{64}$$

Exponential Series

$$1.\quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$2.\ e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$
$$3. e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \dots$$

$$\implies e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

$$\implies \frac{e^x + e^{-x}}{2} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$
$$4. \cosh x = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$
$$5. \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Problem 9. *Show that* $\log 2 - \frac{(\log 2)^2}{2!} + \frac{(\log 2)^3}{3!} \cdots = \frac{1}{2}$

Solution:

$$S = y - \frac{y^2}{2!} + \frac{y^3}{3!} \dots$$
$$1 - S = 1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} \dots$$
$$1 - S = e^{-y}$$
$$\implies 1 - S = e^{-\log 2}$$
$$\implies 1 - S = e^{\log 2^{-1}}$$
$$\implies 1 - S = e^{\log \frac{1}{2}}$$
$$\implies 1 - S = \frac{1}{2}$$
$$\implies S = 1 - \frac{1}{2}$$
$$\implies S = \frac{1}{2}$$

Problem 10. *Prove that*

$$\frac{e^z - 1}{e^z + 1} = \frac{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots}{1 + \frac{1}{2!} + \frac{1}{4!} + \dots}$$

Solution:

$$RHS = \frac{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots}{1 + \frac{1}{2!} + \frac{1}{4!} + \dots} = \frac{\sinh(1)}{\cosh(1)}$$
$$\implies \frac{\frac{e^1 - e^{-1}}{2}}{\frac{e^1 + e^{-1}}{2}} = \frac{e^1 - e^{-1}}{e^1 + e^{-1}} = \frac{e^1 - \frac{1}{e}}{e^1 + \frac{1}{e}}$$
$$\implies \frac{\frac{e^2 - 1}{e}}{\frac{e^2 + 1}{e}} = \frac{e^2 - 1}{e^2 + 1} = LHS$$

Hence,

$$LHS = RHS.$$

Problem 11. *Sum the series*

$$\frac{1^2}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\dots$$

Solution

$$(n+1)(2n+1)=A+B(n+2)+c(n+2)(n+1)$$

put $n=-2$

$$\implies (-2+1)(-4+1)=A+B(0)+c(0)$$
$$A=3$$

put $n=-1$

$$\implies 0=A+B(-1+2)+0$$
$$\implies B=-A$$
$$\implies B=-3$$

Equating the coeffient of n^2, we get

$$c=2$$

Now,

$$t_n=\frac{(n+1)(2n+1)}{(n+2)}=\frac{3-3(n+2)+2(n+2)}{(n+2)!}$$

$$\begin{aligned}\sum_{n=1}^{\infty}t_n&=\sum_{n=1}^{\infty}\frac{1}{(n+2)!}-3\sum_{n=1}^{\infty}\frac{(n+2)}{(n+2)!}+2\sum_{n=1}^{\infty}\frac{(n+2)(n+1)}{(n+2)!}\\&=3\left[\frac{1}{3!}+\frac{1}{6!}+\dots\right]-3\left[\frac{1}{2!}+\frac{1}{3!}+\dots\right]+2\left[\frac{1}{1!}+\frac{1}{2!}+\dots\right]\\&=3\left[e-1-\frac{1}{1!}-\frac{1}{2!}\right]-3\left[e-1-\frac{1}{1!}\right]+2(e-1)\\&=3e-\frac{5}{2}-3+6+2e-2\\&=2e-\frac{7}{2}\end{aligned}$$

Problem 12. *Show that* $\frac{1}{1.1.3} + \frac{1}{2.3.5} + \frac{1}{3.5.7} + \cdots = 2\log 2 - 1$

Solution

$$t_n = \frac{1}{n(2n-1)(2n+1)}$$

Now,

$$\frac{1}{n(2n-1)(2n+1)} = \frac{A}{n} + \frac{B}{2n-1} + \frac{c}{2n+1}$$
$$1 = A(2n-1)(2n+1) + Bn(2n+1) + cn(2n-1).$$

Put $n = \frac{1}{2}$

$$1 = B\frac{1}{2}\left(2\frac{1}{2} + 1\right)$$
$$\implies 1 = B\frac{1}{2}(2)$$
$$\implies B = 1$$

Put $n = 0$

$$1 = A(-1)$$
$$\implies A = -1$$

Put $n = -\frac{1}{2}$

$$1 = c\left(-\frac{1}{2}\right)\left(2\left(-\frac{1}{2}\right) - 1\right)$$
$$\implies c = 1.$$

Now,

$$\begin{aligned}
\sum_{n=1}^{\infty} t_n &= -\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)} \\
&= -\left[1 + \frac{1}{2} + \frac{1}{3} + \dots\right] + \left[1 + \frac{1}{3} + \frac{1}{5} + \dots\right] + \left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\right] \\
&= -\left[\frac{1}{2} + \frac{1}{3} + \dots\right] + \left[\frac{1}{3} + \frac{1}{5} + \dots\right] + \left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\right] \\
&= 2\left[\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\right]\right] - 2\left[\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots\right] \\
&= 2\left[\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right] \\
&= 2\left[\log(1+1) - 1 + \frac{1}{2}\right] \\
&= 2\left[\log 2 - \frac{1}{2}\right] \\
&= 2\log 2 - 1
\end{aligned}$$

Problem 13. *If A and B are the roots of the equation $x^2 + px + q = 0$. Prove that*

$$\log\left(1 - px + qx^2\right) = s_1 x - \frac{1}{2} s_2 x^2 + \frac{1}{3} s_3 x^3$$

where $s_r = A^r + B^r$.

Solution:

$$\begin{aligned}
1 - px + qx^2 &= 1 + (A+B)x + ABx^2 \\
&= 1 + Ax + Bx + ABx^2 \\
&= (1 + Ax) + Bx(1 + Ax) \\
&= (1 + Ax)(1 + Bx).
\end{aligned}$$

Taking log on both sides, we get

$$\begin{aligned}\log(1 - px + qx^2) &= \log(1 + Ax) + \log(1 + Bx)\\ &= \left(Ax - \frac{A^2x^2}{2} + \frac{A^3x^3}{3} - \dots\right) + \left(Bx - \frac{B^2x^2}{2} + \frac{B^3x^3}{3} - \dots\right)\\ &= x(A + B) - \frac{x^2}{2}(A^2 + B^2) + \frac{x^3}{3}(A^3 + B^3) + \dots\\ &= s_1x - s_2\frac{x^2}{2} + s_3\frac{x^3}{3}.\end{aligned}$$

Therefore, LHS = RHS.